

Locally standard torus actions and h' -vectors of simplicial posets

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ABSTRACT. We consider the orbit type filtration on a manifold X with locally standard action of a compact torus and the corresponding homological spectral sequence $(E_X)_{*,*}^r$. If all proper faces of the orbit space $Q = X/T$ are acyclic and free part of the action is trivial, this spectral sequence can be described in full. The ranks of diagonal terms are equal to the h' -numbers of the Buchsbaum simplicial poset S_Q dual to Q . Betti numbers of X depend only on the orbit space Q but not on the characteristic function. If X is a slightly different object, namely the model space $X = (P \times T^n)/\sim$ where P is a cone over Buchsbaum simplicial poset S , we prove that $\dim(E_X)_{p,p}^\infty = h_p''(S)$. This gives a topological evidence for the fact that h'' -numbers of Buchsbaum simplicial posets are nonnegative.

1. Introduction

An action of a compact torus T^n on a smooth compact manifold M of dimension $2n$ is called locally standard if it is locally modeled by the standard representation of T^n on \mathbb{C}^n . The orbit space $Q = M/T^n$ is a manifold with corners. Every manifold with locally standard torus action is equivariantly homeomorphic to the quotient construction $X = Y/\sim$, where Y is a principal T^n -bundle over Q and \sim is an equivalence relation given by a characteristic function on Q (see [14]).

In the case when all faces of the orbit space (including Q itself) are acyclic, Masuda and Panov [10] proved that $H_T^*(M; \mathbb{Z}) \cong \mathbb{Z}[S_Q]$ and $H^*(M; \mathbb{Z}) \cong \mathbb{Z}[S_Q]/(l.s.o.p)$, where S_Q is a simplicial poset dual to Q ; $\mathbb{Z}[S_Q]$ is the face ring with even grading; and $(l.s.o.p)$ is the system of parameters of degree 2 determined by the characteristic function. In this situation S_Q is a Cohen–Macaulay simplicial poset, so $(l.s.o.p)$ is actually a regular sequence in the ring $\mathbb{Z}[S_Q]$. In particular this implies $\dim H^{2j}(M) = h_j(S_Q)$ and $H^{2j+1}(M) = 0$.

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These considerations generalize similar results for quasitoric manifolds, complete smooth toric varieties, and symplectic toric manifolds which were known before. One can see that there are many examples of manifolds M whose orbit spaces are acyclic. Nevertheless, several constructions had appeared in the last years providing natural examples of manifolds with nontrivial topology of the orbit space. These constructions include for example toric origami manifolds [6] and toric log symplectic manifolds [8].

It seems that the most reasonable assumption which is weaker than acyclicity of all faces, but still allows for explicit calculations is the following. We assume that every *proper* face of Q is acyclic, and Y is a trivial T^n -bundle: $Y = Q \times T^n$. This paper is the second in a series of works, where we study homological structure of M under these assumption by using the orbit type filtration. The general scopes of this work are described in the preprint [1].

It is convenient to work with a quotient construction $X = (Q \times T^n)/\sim$ instead of M . The orbit type filtration $X_0 \subset X_1 \subset \dots \subset X_n$ covers the natural filtration $Q_0 \subset Q_1 \subset \dots \subset Q_n$ of Q , and is covered by a filtration $Y_0 \subset Y_1 \subset \dots \subset Y_n$ of Y , where $Y_i = Q_i \times T^n$. In the previous paper [2] we proved that homological spectral sequences associated with filtrations on Y and X are closely related. Namely, there is an isomorphism of the second pages $f_*^2: (E_Y)_{p,q}^2 \rightarrow (E_X)_{p,q}^2$ for $p > q$, when Q has acyclic proper faces.

In this paper we calculate the ranks of groups in the spectral sequence and Betti numbers of X . Since $Y = Q \times T^n$, the spectral sequence $(E_Y)_{*,*}^*$ is isomorphic to $(E_Q)_{*,*}^* \otimes H_*(T^n)$. The structure of $(E_Q)_{*,*}^*$ can be explicitly described. This is done in Section 3. As a technical tool, we introduce the modified spectral sequence $(\dot{E}_Q)^*$ which coincides with $(E_Q)^*$ from the second page, and whose first page $(\dot{E}_Q)_{*,*}^1$ in a certain sense lies in between $(E_Q)_{*,*}^1$ and $(E_Q)_{*,*}^2$. Similar constructions of modified spectral sequences $(\dot{E}_Y)^*$ and $(\dot{E}_X)^*$ are introduced for Y and X in Section 4.

The induced map $f_*^1: (\dot{E}_Y)_{p,q}^1 \rightarrow (\dot{E}_X)_{p,q}^1$ is an isomorphism for $p > q$, as follows essentially from the result of [2]. This gives a description of all differentials and all non-diagonal terms of $(\dot{E}_X)_{*,*}^1$, which is stated in Theorem 1. The diagonal terms of the spectral sequence require an independent investigation. We prove, in particular, that $\dim(E_X)_{q,q}^2 = \dim(\dot{E}_X)_{q,q}^2 = h'_{n-q}(S_Q)$ — the h' -number of the dual simplicial poset (Theorem 3). The proof involves combinatorial computations and is placed in separate Section 5, where we give all necessary definitions from the combinatorial theory of simplicial posets. The appearance of h' -vector in this problem is quite natural. If Q has acyclic proper faces, the dual simplicial poset S_Q is Buchsbaum. Recall that h' -vector is a combinatorial notion, which was specially devised to study the combinatorics of Buchsbaum simplicial complexes.

In Section 6 we introduce the bigraded structure on $H_*(X; \mathbb{k})$ and compute bigraded Betti numbers (Theorem 5). Bigraded Poincare duality easily follows from this computation.

Most of the arguments used for manifolds with locally standard actions, work equally well for the space $X = (P \times T^n)/\sim$ where P is the cone over Buchsbaum simplicial poset equipped with the dual face structure. In this case there holds $\dim(E_X)_{q,q}^\infty = h_q''(S)$ (Theorem 4). Over rational numbers, every simplicial poset admits a characteristic function, therefore our theorem implies $h_q''(S) \geq 0$ for a Buchsbaum simplicial poset S . This result was proved by Novik and Swartz in [12] by a different method.

Both cases, manifolds with acyclic proper faces and cones over Buchsbaum posets, are unified in the notion of Buchsbaum pseudo-cell complex, introduced in Section 2. The technique developed in the paper can be applied to any Buchsbaum pseudo-cell complex.

In the last section we analyze a simple example which shows that without the assumption of proper face acyclicity the problem of computing Betti numbers of X is more complicated. In general, Betti numbers of X may depend not only on the orbit space Q , but also on the characteristic function.

2. Preliminaries

2.1. Coskeleton filtrations and manifolds with corners.

DEFINITION 2.1. *A finite partially ordered set (poset in the following) is called simplicial if there is a minimal element $\hat{0} \in S$ and, for any $I \in S$, the lower order ideal $\{J \in S \mid J \leq I\}$ is isomorphic to the poset of faces of a $(k-1)$ -simplex, for some $k \geq 0$.*

The elements of S are called simplices. The number k in the definition is denoted $|I|$ and called the rank of I . Also set $\dim I = |I| - 1$. A simplex of rank 1 is called a vertex; the set of all vertices is denoted $\text{Vert}(S)$. The *link* of a simplex $I \in S$ is the set $\text{lk}_S I = \{J \in S \mid J \geq I\}$. This set inherits the order relation from S , and $\text{lk}_S I$ is a simplicial poset with respect to this order, with I being the minimal element. Let S' denote the barycentric subdivision of S . By definition, S' is a simplicial complex on the vertex set $S \setminus \{\hat{0}\}$ whose simplices are the ordered chains in $S \setminus \{\hat{0}\}$. The geometric realization of S is the geometric realization of its barycentric subdivision $|S| \stackrel{\text{def}}{=} |S'|$. One can also think of $|S|$ as a CW-complex with simplicial cells (such complexes were called *simplicial cell complexes* in [4]). A poset S is called *pure* if all its maximal elements have equal dimensions. A poset S is pure whenever S' is pure.

Let \mathbb{k} denote a ground ring, which may be either \mathbb{Z} or a field. The term “(co)homology of simplicial poset” means the (co)homology of its geometrical realization. If the coefficient ring in the notation of (co)homology is omitted, it is supposed to be \mathbb{k} . The rank of a \mathbb{k} -module A is denoted $\dim A$.

DEFINITION 2.2. A simplicial poset S of dimension $n - 1$ is called *Buchsbaum* (over \mathbb{k}) if $\tilde{H}_i(\mathrm{lk}_S I; \mathbb{k}) = 0$ for all $\hat{0} \neq I \in S$ and $i \neq n - 1 - |I|$. If S is Buchsbaum and, moreover, $\tilde{H}_i(S; \mathbb{k}) = 0$ for $i \neq n - 1$, then S is called *Cohen–Macaulay* (over \mathbb{k}).

By abuse of terminology we call S a *homology manifold* of dimension $n - 1$ if its geometric realization $|S|$ is a homology $(n - 1)$ -manifold. Simplicial poset S is Buchsbaum if and only if it is Buchsbaum and, moreover, its local homology stack of highest degree is isomorphic to a constant sheaf (see details in [2]).

If S is Buchsbaum and connected, then S is pure. In the following we consider only pure posets, and assume $\dim S = n - 1$.

CONSTRUCTION 2.3. For any pure simplicial poset S , there is an associated space $P(S) = \mathrm{Cone} |S|$ endowed with the dual face structure (also called coskeleton structure), defined as follows. The complex $P(S)$ is a simplicial complex on the set S and k -simplices of S' have the form $(I_0 < I_1 < \dots < I_k)$, where $I_j \in S$. For each $I \in S$ consider the subsets:

$$G_I = |\{(I_0 < I_1 < \dots) \in S' \text{ such that } I_0 \geq I\}| \subset P(S),$$

$$\partial G_I = |\{(I_0 < I_1 < \dots) \in S' \text{ such that } I_0 > I\}| \subset P(S).$$

and the subset $G_I^\circ = G_I \setminus \partial G_I$. We have $G_{\hat{0}} = P(S)$; $G_I \subset G_J$ whenever $J < I$, and $\dim G_I = n - 1 - \dim I$ since S is pure. A subset G_I is called a *dual face* of a simplex $I \in S$. A subset ∂G_I is a union of faces of smaller dimensions.

Recall several facts about manifolds with corners. A smooth connected manifold with corners Q is called *nice* (or a *manifold with faces*) if every codimension k face lies in exactly k distinct facets. In the following we consider only nice compact orientable manifolds with corners. Any such Q determines a simplicial poset S_Q whose elements are the faces of Q ordered by reversed inclusion. The whole Q is the maximal face of itself, thus represents the minimal element of S_Q .

DEFINITION 2.4. A nice manifold with corners Q is called *Buchsbaum* if Q is orientable and every proper face of Q is acyclic. If, moreover, Q is acyclic itself, it is called *Cohen–Macaulay*.

If Q is a Buchsbaum manifold with corners, then its underlying simplicial poset S_Q is Buchsbaum (moreover, S_Q is a homology manifold), and when Q is Cohen–Macaulay, then so is S_Q (moreover, S_Q is a homology sphere) by [2, Lm.6.2].

2.2. Buchsbaum pseudo-cell complexes. It is convenient to introduce a notion which captures both manifolds with corners and cones over simplicial posets.

CONSTRUCTION 2.5 (Pseudo-cell complex). A CW-pair $(F, \partial F)$ will be called k -dimensional pseudo-cell, if F is compact and connected, $\dim F = k$, $\dim \partial F \leq k - 1$. A (regular finite) *pseudo-cell complex* Q is a space which is a union of an expanding sequence of subspaces Q_k such that Q_{-1} is empty and Q_k is the pushout obtained

from Q_{k-1} by attaching finite number of k -dimensional pseudo-cells $(F, \partial F)$ along injective attaching CW-maps $\partial F \rightarrow Q_{k-1}$. We also assume that the boundary of each pseudo-cell is a union of lower dimensional pseudo-cells. The poset of pseudo-cells, ordered by the reversed inclusion is denoted by S_Q . The abstract elements of S_Q are denoted by I, J , etc. and the corresponding pseudo-cells considered as subsets of Q are denoted F_I, F_J , etc.

A pseudo-cell complex Q , of dimension n is called *simple* if S_Q is a simplicial poset of dimension $n - 1$ and $\dim F_I = n - 1 - \dim I$ for all $I \in S_Q$. In particular, the space Q itself represents the maximal pseudo-cell, $Q = F_{\hat{0}}$. Pseudo-cells of a simple pseudo-cell complex Q will be called *faces*, faces different from Q — *proper faces*, and maximal proper faces — *facets*. Facets correspond to vertices of S_Q .

Examples of simple pseudo-cell complexes are nice manifolds with corners and cones over simplicial posets. Simple polytopes are examples, which lie in both of these classes.

DEFINITION 2.6. *A simple pseudo-cell complex Q is called Buchsbaum (over \mathbb{k}) if, for any proper face $F_I \subset Q$, $I \neq \hat{0}$, the following conditions hold:*

- (1) F_I is acyclic, $\tilde{H}_*(F_I; \mathbb{k}) = 0$;
- (2) $H_j(F_I, \partial F_I; \mathbb{k}) = 0$ for each $j \neq \dim F_I$.

Buchsbaum complex Q is called Cohen–Macaulay (over \mathbb{k}) if these two conditions also hold for the maximal face $F_{\hat{0}} = Q$.

Both Buchsbaum manifolds and cones over Buchsbaum posets are examples of Buchsbaum pseudo-cell complexes (and the same for Cohen–Macaulay property). Indeed, in the case of Buchsbaum manifold with corners, $\tilde{H}_*(F_I)$ vanishes by definition and $H_*(F_I, \partial F_I)$ vanishes in the required degrees by the Poincaré–Lefschetz duality, since every face F_I is an orientable manifold with boundary. In the cone case, we have $G_I = \text{Cone}(\partial G_I)$ and $\partial G_I \cong |\text{lk}_S I|$, so the conditions of Definition 2.6 follow from the isomorphism $H_*(G_I, \partial G_I) \cong H_{*-1}(\partial G_I)$ and Definition 2.2.

We have a topological filtration

$$Q_0 \subset Q_1 \subset \dots \subset Q_{n-1} \subset Q_n = Q;$$

and a truncated filtration

$$Q_0 \subset Q_1 \subset \dots \subset Q_{n-1} = \partial Q,$$

where Q_j is a union of faces of dimension $\leq j$. The homological spectral sequences associated with these filtrations are denoted $(E_Q)_{p,q}^r$ and $(E_{\partial Q})_{p,q}^r$ respectively. The same argument as in [2, Lm.6.2] proves the following

PROPOSITION 2.7.

- (1) *Let Q be a Buchsbaum pseudo-cell complex, S_Q be its underlying poset, and $P = P(S_Q)$ be the cone complex. Then there exists a face-preserving map $\varphi: Q \rightarrow P$ which induces the identity isomorphism of posets of faces and*

an isomorphism of the truncated spectral sequences $\varphi_*: (E_{\partial Q})_{*,*}^r \xrightarrow{\cong} (E_{\partial P})_{*,*}^r$ for $r \geq 1$. In particular, if Q is a Buchsbaum pseudo-cell complex, then S_Q is a Buchsbaum simplicial poset.

- (2) If Q is a Cohen–Macaulay pseudo-cell complex of dimension n , then φ induces an isomorphism of non-truncated spectral sequences $\varphi_*: (E_Q)_{*,*}^r \xrightarrow{\cong} (E_P)_{*,*}^r$ for $r \geq 1$. In particular, if Q is a Cohen–Macaulay pseudo-cell complex, then S_Q is a Cohen–Macaulay simplicial poset.

Thus all homological information about Buchsbaum pseudo-cell complex Q away from its maximal cell is encoded in the underlying poset S_Q . This makes Buchsbaum pseudo-cell complexes, and in particular Buchsbaum manifolds with corners, a good family to study.

3. Spectral sequence of Q

3.1. Truncated and non-truncated spectral sequences. In Buchsbaum case the spectral sequence $(E_Q)_{p,q}^r$ can be described explicitly. We have $(E_Q)_{p,q}^r \Rightarrow H_{p+q}(Q)$, the differentials act as $(d_Q)^r: (E_Q)_{p,q}^r \rightarrow (E_Q)_{p-r,q+r-1}^r$, and

$$(E_Q)_{p,q}^1 \cong H_{p+q}(Q_p, Q_{p-1}) \cong \bigoplus_{I, \dim F_I = p} H_{p+q}(F_I, \partial F_I).$$

By the definition of Buchsbaum pseudo-cell complex, we have $(E_Q)_{p,q}^1 = 0$ unless $q = 0$ or $p = n$. Such form of the spectral sequence will be referred to as \sqcap -shaped.

By forgetting the last term of the filtration we get the spectral sequence $(E_{\partial Q})_{p,q}^r \Rightarrow H_{p+q}(\partial Q)$, whose terms vanish unless $q = 0$. Thus $(E_{\partial Q})_{p,q}^r$ collapses at a second page, giving the isomorphism $(E_{\partial Q})_{p,0}^2 \cong H_p(\partial Q)$.

In the non-truncated case we have $(E_Q)_{p,0}^2 \cong (E_{\partial Q})_{p,0}^2$ for $p \neq n, n-1$. The terms $(E_Q)_{n,q}^2$ coincide with $(E_Q)_{n,q}^1 \cong H_{n+q}(Q, \partial Q)$ when $q \neq 0$. The term $(E_Q)_{n-1,0}^2$ differs from $(E_{\partial Q})_{n-1,0}^2 \cong H_{n-1}(\partial Q)$ by the image of the first differential $(d_Q)^1$ which hit it at the previous step. Similarly, the term $(E_Q)_{n,0}^2$ is the kernel of the same differential. To avoid mentioning these two exceptional cases every time in the following, we introduce the formalism of modified spectral sequence.

3.2. Modified spectral sequence. Let $(\dot{E}_Q)_{*,*}^1$ be the collection of \mathbb{k} -modules defined by

$$(\dot{E}_Q)_{p,q}^1 \stackrel{\text{def}}{=} \begin{cases} (E_{\partial Q})_{p,q}^2, & \text{if } p \leq n-1, \\ (E_Q)_{p,q}^1, & \text{if } p = n, \\ 0, & \text{otherwise.} \end{cases}$$

Let d_Q^- be the differential of degree $(-1, 0)$ acting on $\bigoplus (E_Q)_{p,q}^1$ by:

$$d_Q^- = \begin{cases} (d_Q)^1: (E_Q)_{p,q}^1 \rightarrow (E_Q)_{p-1,q}^1, & \text{if } p \leq n-1, \\ 0, & \text{otherwise} \end{cases}$$

It is easily seen that the homology module $H((E_Q)^1; d_Q^-)$ is isomorphic to $(\dot{E}_Q)^1$. Now consider the differential $(\dot{d}_Q)^1$ of degree $(-1, 0)$ acting on $\bigoplus (\dot{E}_Q)^1_{p,q}$:

$$(\dot{d}_Q)^1 = \begin{cases} 0, & \text{if } p \leq n-1; \\ (E_Q)^1_{n,q} \xrightarrow{(d_Q)^1} (E_Q)^1_{n-1,q}, & \text{if } p = n. \end{cases}$$

In the latter case, the image of the differential lies in $(\dot{E}_Q)^1_{n-1,q} \subseteq (E_Q)^1_{n-1,q}$ since $(\dot{E}_Q)^1_{n-1,q}$ is just the kernel of $(d_Q)^1$. We have $(E_Q)^2 \cong H((E_Q)^1, (\dot{d}_Q)^1)$. These considerations are shown on the diagram:

$$\begin{array}{ccccc} & & (\dot{E}_Q)^1 & & \\ & \nearrow d_Q^- & & \searrow (\dot{d}_Q)^1 & \\ (E_Q)^1 & \xrightarrow{(d_Q)^1} & (E_Q)^2 & \xrightarrow{(d_Q)^2} & (E_Q)^3 \xrightarrow{(d_Q)^3} \dots \end{array}$$

in which the dotted arrows represent passing to homology. To summarize:

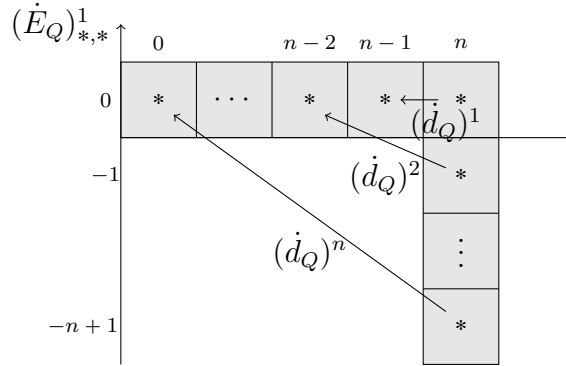


FIGURE 1. The shape of the spectral sequence.

CLAIM 3.1. *There is a homological spectral sequence $(\dot{E}_Q)_{p,q}^r \Rightarrow H_{p+q}(Q)$ such that $(\dot{E}_Q)_{*,*}^1 = H((E_Q)^1, d_Q^-)$, and $(\dot{E}_Q)_{*,*}^r = (E_Q)_{*,*}^r$ for $r \geq 2$. The only nontrivial differentials of this sequence have the form*

$$(\dot{d}_Q)^r: (\dot{E}_Q)_{n,1-r}^r \rightarrow (\dot{E}_Q)_{n-r,0}^r$$

for $r \geq 1$ (see Figure 1).

The differentials have pairwise distinct domains and targets. Thus the whole spectral sequence $(E_Q)_{p,q}^r \Rightarrow H_{p+q}(Q)$ folds into a single long exact sequence, which is isomorphic to a long exact sequence of the pair $(Q, \partial Q)$:

(3.1)

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & H_{n+1-r}(Q) & \longrightarrow & (\dot{E}_Q)_{n,1-r}^r & \xrightarrow{(\dot{d}_Q)^r} & (\dot{E}_Q)_{n-r,0}^r \longrightarrow H_{n-r}(Q) \longrightarrow \dots \\
 & & \parallel & & \parallel & & \parallel \\
 & & & & (\dot{E}_Q)_{n,1-r}^1 & & (\dot{E}_Q)_{n-r,0}^1 \\
 & & & & \parallel & & \downarrow \cong \\
 \dots & \longrightarrow & H_{n+1-r}(Q) & \longrightarrow & H_{n+1-r}(Q, \partial Q) & \xrightarrow{\delta_{n+1-r}} & H_{n-r}(\partial Q) \longrightarrow H_{n-r}(Q) \longrightarrow \dots
 \end{array}$$

In particular, the differentials $(d_Q)^r: (E_Q)_{n,1-r}^r \rightarrow (E_Q)_{n-r,0}^r$ coincide up to isomorphism with the connecting homomorphisms

$$\delta_{n+1-r}: H_{n+1-r}(Q, \partial Q) \rightarrow H_{n-r}(\partial Q).$$

This proves

PROPOSITION 3.2. *Up to isomorphism, the spectral sequence $(\dot{E}_Q)_{*,*}^r \Rightarrow H_*(Q)$ has the form:*

The diagram illustrates a spectral sequence with a grid of terms. The horizontal axis represents the filtration index, with labels 0, $n-2$, $n-1$, and n . The vertical axis represents the spectral index, with labels 0, -1 , and $-n+1$. The top row (index 0) contains terms $H_0(\partial Q)$, \dots , $H_{n-2}(\partial Q)$, $H_{n-1}(\partial Q)$, and $H_n(Q, \partial Q)$. The bottom row (index $-n+1$) contains $H_1(Q, \partial Q)$. Arrows indicate differentials: $(\dot{d}_Q)^1 = \delta_n$ from $H_n(Q, \partial Q)$ to $H_{n-1}(\partial Q)$, $(\dot{d}_Q)^2 = \delta_{n-1}$ from $H_{n-1}(Q, \partial Q)$ to $H_{n-2}(\partial Q)$, and $(\dot{d}_Q)^n = \delta_1$ from $H_1(Q, \partial Q)$ to $H_0(\partial Q)$.

4. Quotient construction and its spectral sequence

4.1. Quotient construction. Let T^n denote a compact torus, and Λ_* be its homology algebra, $\Lambda_* = \bigoplus_{j=0}^n \Lambda_j$, $\Lambda_j = H_j(T^n; \mathbb{k})$. Let Q be a simple pseudo-cell complex of dimension n , and S_Q its dual simplicial poset. The map

$$\lambda: \text{Vert}(S_Q) \rightarrow \{1\text{-dimensional toric subgroups of } T^n\}$$

is called *characteristic function* if the following so called $(*)$ -condition holds: whenever i_1, \dots, i_k are the vertices of some simplex in S_Q , the map

$$(4.1) \quad \lambda(i_1) \times \dots \times \lambda(i_k) \rightarrow T^n,$$

induced by inclusions $\lambda(i_j) \hookrightarrow T^n$, is injective and splits. Note that i_1, \dots, i_k are the vertices of some simplex, if and only if $F_{i_1} \cap \dots \cap F_{i_k} \neq \emptyset$. Denote the image of the map (4.1) by T_I , where I is a simplex with vertices i_1, \dots, i_k .

It follows from the $(*)$ -condition that the map

$$(4.2) \quad H_1(\lambda(F_1) \times \dots \times \lambda(F_k); \mathbb{k}) \rightarrow H_1(T^n; \mathbb{k})$$

is injective and splits for every \mathbb{k} . If the map (4.2) splits for a specific ground ring \mathbb{k} , we say that λ satisfies $(*_\mathbb{k})$ -condition and call it a \mathbb{k} -characteristic function. It is easy to see that the topological $(*)$ -condition is equivalent to $(*_\mathbb{Z})$, and that $(*_\mathbb{Z})$ implies $(*_\mathbb{k})$ for any \mathbb{k} .

For a simple pseudo-cell complex Q of dimension n , consider the space $Y = Q \times T^n$.

CONSTRUCTION 4.1. For any \mathbb{k} -characteristic function λ over Q consider the quotient construction

$$X = Y / \sim = (Q \times T^n) / \sim,$$

where $(q_1, t_1) \sim (q_2, t_2)$ if and only if $q_1 = q_2 \in F_I^\circ$ for some $I \in S_Q$ and $t_1 t_2^{-1} \in T_I$. The action of T^n on the second coordinate of Y descends to the action on X . The orbit space of this action is Q . The stabilizer of the point $q \in F_I^\circ \subset Q$ is T_I . Let f denote the canonical quotient map from Y to X .

The filtration on Q induces filtrations on Y and X :

$$Y_i = Q_i \times T^n, \quad X_i = Y_i / \sim, \quad i = 0, \dots, n.$$

The filtration $X_0 \subset X_1 \subset \dots \subset X_n = X$ coincides with the orbit type filtration. This means that X_i is a union of all torus orbits of dimension at most i . We have $\dim X_i = 2i$. We will use the following notation

$$\begin{aligned} Y_I &= F_I \times T^n, & \partial Y_I &= (\partial F_I) \times T^n \\ X_I &= Y_I / \sim, & \partial X_I &= \partial Y_I / \sim \end{aligned}$$

for $I \in S_Q$. Note that ∂X_I does not have the meaning of topological boundary of X_I , this is just a conventional notation. Since $\partial Q = Q_{n-1}$, we have $\partial Y = Y_{n-1} = (\partial Q) \times T^n$ and $\partial X = X_{n-1} = \partial Y / \sim$.

There are homological spectral sequences

$$\begin{aligned} (E_Y)_{p,q}^r &\Rightarrow H_{p+q}(Y) & (E_X)_{p,q}^r &\Rightarrow H_{p+q}(X) \\ (E_{\partial Y})_{p,q}^r &\Rightarrow H_{p+q}(\partial Y) & (E_{\partial X})_{p,q}^r &\Rightarrow H_{p+q}(\partial X), \end{aligned}$$

associated with these filtrations. The canonical map $f: Y \rightarrow X$ induces the morphisms $f_*^r: (E_Y)_{*,*}^r \rightarrow (E_X)_{*,*}^r$ and $f_*^r: (E_{\partial Y})_{*,*}^r \rightarrow (E_{\partial X})_{*,*}^r$.

Since homology groups of the torus are torsion free, we have

$$(E_Y)_{p,q}^r \cong \bigoplus_{q_1+q_2=q} (E_Q)_{p,q_1}^r \otimes \Lambda_{q_2},$$

for $r \geq 1$ by Kunneth's formula. Similar for the truncated spectral sequence:

$$(E_{\partial Y})_{p,q}^r \cong \bigoplus_{q_1+q_2=q} (E_{\partial Q})_{p,q_1}^r \otimes \Lambda_{q_2},$$

4.2. Modified spectral sequences. As in the case of Q and absolutely similar to that case, we introduce the modified spectral sequences $(\dot{E}_Y)_{*,*}$ and $(\dot{E}_X)_{*,*}$. Consider the bigraded module:

$$(\dot{E}_Y)_{p,q}^1 = \begin{cases} (E_{\partial Y})_{p,q}^2, & \text{if } p < n; \\ (E_Y)_{n,q}^1, & \text{if } p = n. \end{cases}$$

and define the differentials $d_Y^-: (E_Y)_{p,q}^1 \rightarrow (E_Y)_{p-1,q}^1$ and $(\dot{d}_Y)^1: (\dot{E}_Y)_{p,q}^1 \rightarrow (\dot{E}_Y)_{p-1,q}^1$ by

$$d_Y^- = \begin{cases} (d_Y)^1, & \text{if } p < n; \\ 0, & \text{if } p = n. \end{cases} \quad (\dot{d}_Y)^1 = \begin{cases} 0, & \text{if } p < n; \\ (E_Y)_{n,q}^1 \xrightarrow{(d_Y)^1} (E_Y)_{n-1,q}^1 & \text{if } p = n. \end{cases}$$

It is easily checked that $(\dot{E}_Y)^1 \cong H((E_Y)^1, d_Y^-)$ and $(E_Y)^2 \cong H((\dot{E}_Y)^1, (\dot{d}_Y)^1)$. Let $(\dot{E}_Y)^r = (E_Y)^r$ for $r \geq 2$. Thus we have the modified spectral sequence $(\dot{E}_Y)_{*,*}^r \Rightarrow H_*(Y)$. For $r \geq 1$ we have

$$(4.3) \quad (\dot{E}_Y)_{p,q}^r \cong \bigoplus_{q_1+q_2=q} (\dot{E}_Q)_{p,q_1}^r \otimes \Lambda_{q_2},$$

The same construction applies for X , thus we get the spectral sequence $(\dot{E}_X)_{*,*}^r \Rightarrow H_*(X)$ such that $(\dot{E}_X)^1 \cong H((E_X)^1, d_X^-)$, and $(\dot{E}_X)^r = (E_X)^r$ for $r \geq 2$. There exists an induced map of the modified spectral sequences:

$$\dot{f}_*^r: (\dot{E}_Y)^r \rightarrow (\dot{E}_X)^r.$$

By dimensional reasons the homological spectral sequence $(E_X)_{p,q}^r \Rightarrow H_{p+q}(X)$ (and therefore its modified version) has an obvious vanishing property:

$$(E_X)_{p,q}^1 = H_{p+q}(X_p, X_{p-1}) = 0 \text{ for } q > p.$$

PROPOSITION 4.2. *The map $\dot{f}_*^1: (\dot{E}_Y)_{p,q}^1 \rightarrow (\dot{E}_X)_{p,q}^1$ is an isomorphism when $p > q$ or $p = q = n$. It is injective when $p = q$. In other cases, i.e. when $p < q$, the modules $(\dot{E}_X)_{p,q}^1$ vanish.*

PROOF. The map $\dot{f}_*^2: (E_{\partial Y})_{p,q}^2 \rightarrow (E_{\partial X})_{p,q}^2$ is an isomorphism for $p > q$ and injective for $p = q$ (see [2, Th.3 and Remark 6.6]). Thus $\dot{f}_*^1: (\dot{E}_Y)_{p,q}^1 \rightarrow (\dot{E}_X)_{p,q}^1$ is an isomorphism for $q < p < n$ and injective for $q = p < n$. Note that in [2] we

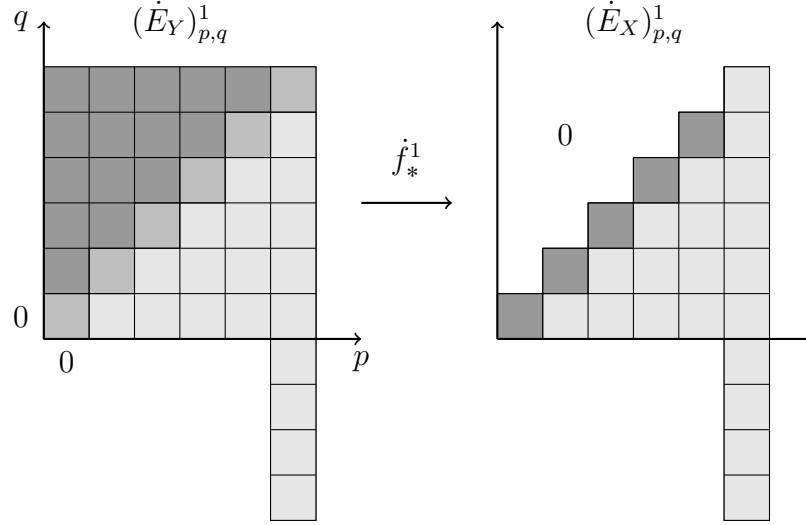


FIGURE 2. The induced map of spectral sequences is an isomorphism below the diagonal and injective on the diagonal.

considered manifolds with corners, but the argument used there can be applied to any Buchsbaum pseudo-cell complex without significant changes.

As for the case $p = n$, the map $f_*: (E_Y)_{n,q}^1 \rightarrow (E_X)_{n,q}^1$ is an isomorphism since the identification \sim does not touch the interior of Y and, therefore,

$$X_n/X_{n-1} = X/\partial X \cong Y/\partial Y = Y_n/Y_{n-1}.$$

Thus $f_*^1: (\dot{E}_Y)_{n,q}^1 = H_{n+q}(Y, \partial Y) \rightarrow (\dot{E}_X)_{n,q}^1 = H_{n+q}(X, \partial X)$ is an isomorphism by excision. \square

This proposition together with (4.3) and Proposition 3.2 gives a complete description of differentials and non-diagonal terms of $(\dot{E}_X)_{*,*}^r$.

THEOREM 1. *Let Q be a Buchsbaum (over \mathbb{k}) pseudo-cell complex, and let $X = (Q \times T^n)/\sim$ be the quotient construction determined by some \mathbb{k} -characteristic function on Q . There exists a homological spectral sequence $(\dot{E}_X)_{*,*}^r$ converging to $H_*(X)$. From its second page this spectral sequence coincides with $(E_X)_{*,*}^*$, the spectral sequence associated with the orbit type filtration. The first page, $(\dot{E}_X)^1$ is the homology module of $(E_X)^1$ with respect to the differential d_X^- of degree $(-1, 0)$. The following properties hold for $(\dot{E}_X)_{*,*}^*$:*

(1) *Non-diagonal terms of the first page have the form*

$$(\dot{E}_X)_{p,q}^1 \cong \begin{cases} H_p(\partial Q) \otimes \Lambda_q, & \text{if } q < p < n; \\ \bigoplus_{q_1+q_2=q+n} H_{q_1}(Q, \partial Q) \otimes \Lambda_{q_2}, & \text{if } p = n; \\ 0, & \text{if } q > p; \end{cases}$$

(2) *There exist injective maps $f_*^1: H_q(\partial Q) \otimes \Lambda_q \hookrightarrow (\dot{E}_X)_{q,q}^1$ for $q \leq n$.*

(3) *Nontrivial differentials for $r \geq 1$ have the form*

$$(\dot{d}_X)^r \cong \begin{cases} \begin{matrix} (\dot{E}_X)_{n, q_1+q_2-n}^1 & (\dot{E}_X)_{q_1-1, q_2}^1 \\ \delta_{q_1} \otimes \text{id}_\Lambda: H_{q_1}(Q, \partial Q) \otimes \Lambda_{q_2} \rightarrow H_{q_1-1}(\partial Q) \otimes \Lambda_{q_2}, & \text{if } r = n - q_1 + 1, q_1 - 1 > q_2; \\ f_*^1 \circ (\delta_{q_1} \otimes \text{id}_\Lambda): H_{q_1}(Q, \partial Q) \otimes \Lambda_{q_2} \rightarrow H_{q_1-1}(\partial Q) \otimes \Lambda_{q_2} \hookrightarrow (\dot{E}_X)_{q_1-1, q_1-1}^*, & \text{if } r = n - q_1 + 1, q_1 - 1 = q_2; \end{matrix} \\ 0, & \text{otherwise.} \end{cases}$$

4.3. Diagonal terms of the spectral sequence. Our next goal is to compute the diagonal terms $(\dot{E}_X)_{q,q}^1$, since they are not described explicitly by Theorem 1. In this subsection we state the results about the dimensions of these modules. The proofs are given in the next section.

Let $\tilde{\beta}_p(S) = \dim \tilde{H}_p(S)$ for $p < n$. If Q is a Buchsbaum pseudo-cell complex, we have $\dim \tilde{H}_p(\partial Q) = \tilde{\beta}_p(S_Q)$, since S_Q is homologous to ∂Q by Proposition 2.7. Let $h_q(S)$, $h'_q(S)$, and $h''_q(S)$ be the h -, h' -, and h'' -numbers of a simplicial poset S (see definitions in Section 5).

THEOREM 2. *In the notation and under conditions of Theorem 1 there holds*

$$\dim(\dot{E}_X)_{q,q}^1 = h_q(S_Q) + \binom{n}{q} \sum_{p=0}^q (-1)^{p+q} \tilde{\beta}_p(S_Q)$$

for $q \leq n-1$.

THEOREM 3. *Let Q be a Buchsbaum manifold with corners and $X = (Q \times T^n)/\sim$. Then:*

- (1) $\dim(\dot{E}_X)_{q,q}^1 = h'_{n-q}(S_Q)$ for $q \leq n-2$, and $\dim(\dot{E}_X)_{n-1, n-1}^1 = h'_1(S_Q) + n$.
- (2) $\dim(\dot{E}_X)_{q,q}^2 = \dim(\dot{E}_X)_{q,q}^1 = h'_{n-q}(S_Q)$ for $0 \leq q \leq n$.

For the cone over Buchsbaum simplicial poset, the diagonal components of ∞ -page also have a clear combinatorial meaning.

THEOREM 4. *Let S be a Buchsbaum simplicial poset, $P = P(S)$ be the cone over its geometric realization, and $X = (P \times T^n)/\sim$. Then*

$$\dim(\dot{E}_X)_{q,q}^\infty = \dim(\dot{E}_X)_{q,q}^\infty = h''_q(S)$$

for $0 \leq q \leq n$.

COROLLARY 4.3. *If S is Buchsbaum, then $h_i''(S) \geq 0$.*

PROOF. For any simplicial poset S there exists a characteristic function on $P = P(S)$ over rational numbers. Thus we can consider the space $X = (P \times T^n)/\sim$ and apply Theorem 4. \square

5. Face vectors and ranks of diagonal components

In this section we prove Theorems 2, 3 and 4.

5.1. Preliminaries on face vectors. First recall several standard definitions from combinatorial theory of simplicial posets.

CONSTRUCTION 5.1. Let S be a pure simplicial poset, $\dim S = n - 1$. Let $f_i(S)$ be the number of i -dimensional simplices in S and, in particular, $f_{-1}(S) = 1$ (the element $\hat{0} \in S$ has dimension -1). The array $(f_{-1}, f_0, \dots, f_{n-1})$ is called the f -vector of S . We write f_i instead of $f_i(S)$ because the poset is always clear from the context. Let $f_S(t)$ be the generating polynomial: $f_S(t) = \sum_{i \geq 0} f_{i-1} t^i$.

Define h -numbers by the relation:

$$(5.1) \quad \sum_{i=0}^n h_i t^i = \sum_{i=0}^n f_{i-1} t^i (1-t)^{n-i} = (1-t)^n f_S\left(\frac{t}{1-t}\right).$$

Let $\beta_i(S) = \dim H_i(S)$, $\tilde{\beta}_i(S) = \dim \tilde{H}_i(S)$, and

$$\chi(S) = \sum_{i=0}^{n-1} (-1)^i \beta_i(S) = \sum_{i=0}^{n-1} (-1)^i f_i(S)$$

$$\tilde{\chi}(S) = \sum_{i=0}^{n-1} \tilde{\beta}_i(S) = \chi(S) - 1.$$

Thus $f_S(-1) = 1 - \chi(S)$. Note that

$$(5.2) \quad h_n = (-1)^{n-1} \tilde{\chi}(S).$$

Define h' - and h'' -numbers of S by the formulas

$$h'_i = h_i + \binom{n}{i} \left(\sum_{j=1}^{i-1} (-1)^{i-j-1} \tilde{\beta}_{j-1}(S) \right) \text{ for } 0 \leq i \leq n;$$

$$h''_i = h'_i - \binom{n}{i} \tilde{\beta}_{i-1}(S) = h_i + \binom{n}{i} \left(\sum_{j=1}^i (-1)^{i-j-1} \tilde{\beta}_{j-1}(S) \right) \text{ for } 0 \leq i \leq n-1,$$

and $h''_n = h'_n$. The summation over an empty set is assumed to be zero. From (5.2) there follows

$$(5.3) \quad h'_n = h_n + \sum_{j=0}^{n-1} (-1)^{n-j-1} \tilde{\beta}_{j-1}(S) = \tilde{\beta}_{n-1}(S).$$

STATEMENT 5.2 (Dehn–Sommerville relations). *For a homology manifold S there holds*

$$(5.4) \quad h_i = h_{n-i} + (-1)^i \binom{n}{i} (1 - (-1)^n - \chi(S)),$$

or, equivalently:

$$(5.5) \quad h_i = h_{n-i} + (-1)^i \binom{n}{i} (1 + (-1)^n \tilde{\chi}(S)).$$

Moreover, $h_i'' = h_{n-i}''$.

PROOF. The first statement can be found in e.g. [13] or [5, Thm.3.8.2]. Also see Remark 5.5 below. The last statement follows from the definition of h'' -vector and Poincare duality $\beta_i(S) = \beta_{n-1-i}(S)$ (see [11, Lm.7.3]). \square

Now we introduce an auxiliary numerical characteristic of a simplicial poset S .

DEFINITION 5.3. *Let S be a Buchsbaum simplicial poset. For $i \geq 0$ consider the number*

$$\hat{f}_i(S) = \sum_{I \in S, \dim I = i} \dim \tilde{H}_{n-1-|I|}(\mathrm{lk}_S I).$$

For a homology manifold S there holds $\hat{f}_i = f_i$ since all proper links are homology spheres. In general, there is another formula connecting these quantities.

PROPOSITION 5.4. *For Buchsbaum simplicial poset S there holds*

$$f_S(t) = (1 - \chi(S)) + (-1)^n \sum_{k \geq 0} \hat{f}_k(S) \cdot (-t - 1)^{k+1}.$$

PROOF. This follows from the general statement [9, Th.9.1], [5, Th.3.8.1], but we provide an independent proof for completeness. As stated in [3, Lm.3.7, 3.8] for simplicial complexes (and not difficult to prove for simplicial posets):

$$\frac{d}{dt} f_S(t) = \sum_{v \in \mathrm{Vert}(S)} f_{\mathrm{lk}_v}(t),$$

and, more generally,

$$\left(\frac{d}{dt} \right)^k f_S(t) = k! \sum_{I \in S, |I|=k} f_{\mathrm{lk}_I}(t).$$

Thus for $k \geq 1$:

$$\begin{aligned} f_S^{(k)}(-1) &= k! \sum_{I \in S, |I|=k} (1 - \chi(\mathrm{lk}_S I)) = \\ &= k! \sum_{I \in S, |I|=k} (-1)^{n-|I|} \dim \tilde{H}_{n-|I|-1}(\mathrm{lk}_S I) = (-1)^{n-k} k! \hat{f}_{k-1}. \end{aligned}$$

The Taylor expansion of $f_S(t)$ at the point -1 has the form:

$$f_S(t) = f_S(-1) + \sum_{k \geq 1} \frac{1}{k!} f_S^{(k)}(-1)(t+1)^k = (1 - \chi(S)) + \sum_{k \geq 0} (-1)^{n-k-1} \hat{f}_k \cdot (t+1)^{k+1}.$$

This finishes the proof. \square

REMARK 5.5. If S is a homology manifold, then Proposition 5.4 implies

$$f_S(t) = (1 - (-1)^n - \chi(S)) + (-1)^n f_S(-t-1),$$

which is yet another equivalent form of Dehn–Sommerville relations (5.4).

LEMMA 5.6. *For Buchsbaum poset S there holds*

$$\sum_{i=0}^n h_i t^i = (1-t)^n (1 - \chi(S)) + \sum_{k \geq 0} \hat{f}_k \cdot (t-1)^{n-k-1}.$$

PROOF. Substitute $t/(1-t)$ in Proposition 5.4 and apply (5.1). \square

Comparing the coefficients at t^i in the identity of Lemma 5.6 we get:

$$(5.6) \quad h_i(S) = (1 - \chi(S))(-1)^i \binom{n}{i} + \sum_{k \geq 0} (-1)^{n-k-i-1} \binom{n-k-1}{i} \hat{f}_k(S).$$

5.2. Ranks of $(E_X)_{*,*}^1$. To prove Theorem 2 we use the following straightforward idea. The module $(\dot{E}_X)^1$ is the homology of $(E_X)^1$ with respect to the differential d_X^- of degree $(-1, 0)$. Theorem 1 describes the ranks of all groups $(\dot{E}_X)_{p,q}^1$ except for $p = q$; the terms $(E_X)_{p,q}^1$ are known as well. Thus the ranks of the remaining terms $\dim(\dot{E}_X)_{q,q}^1$ can be found from the equality of Euler characteristics, computed for $(E_X)^1$ and $(\dot{E}_X)^1$. When we pass from $(E_X)^1$ to $(\dot{E}_X)^1$, the terms with $p = n$ do not change; the other groups are the same as if we passed from $(E_{\partial X})^1$ to $(E_{\partial X})^2$. Thus it is sufficient to perform calculations with the truncated sequence $(E_{\partial X})^*$.

Let χ_q^1 be the Euler characteristic of the q -th row of $(E_{\partial X})_{*,*}^1$:

$$(5.7) \quad \chi_q^1 = \sum_{p \leq n-1} (-1)^p \dim(E_{\partial X})_{p,q}^1.$$

LEMMA 5.7. *For $q \leq n-1$ we have $\chi_q^1 = (\chi(S_Q) - 1) \binom{n}{q} + (-1)^q h_q(S_Q)$.*

PROOF. By Proposition 2.7 there is an isomorphism of spectral sequences $(E_{\partial Q})^* \rightarrow (E_{\partial P(S_Q)})^*$. Thus, in particular, for any $I \in S_Q \setminus \{\hat{0}\}$, $|I| = n - p$ we have an isomorphism

$$(5.8) \quad H_p(F_I, \partial F_I) \cong H_p(G_I, \partial G_I) \cong H_{p-1}(\mathrm{lk}_{S_Q} I)$$

where G_I is the face of $P(S_Q)$ dual to I . The last isomorphism in (5.8) is due to the long exact sequence of the pair $(G_I, \partial G_I)$, since $G_I = \mathrm{Cone}(\partial G_I)$ and $\partial G_I \cong \mathrm{lk}_{S_Q} I$.

For $p < n$ we have

$$\begin{aligned} \dim(E_{\partial X})_{p,q}^1 &= \sum_{I, \dim F_I = p} \dim H_{p+q}(X_I, \partial X_I) = \\ &= \sum_{I, |I| = n-p} \dim(H_p(F_I, \partial F_I) \otimes H_q(T^n/T_I)) = \binom{p}{q} \cdot \hat{f}_{n-p-1}(S_Q). \end{aligned}$$

In the last equality we used (5.8) and the definition of \hat{f} -numbers. Therefore,

$$(5.9) \quad \chi_q^1 = \sum_{p \leq n-1} (-1)^p \dim(E_{\partial X})_{p,q}^1 = \sum_{p \leq n-1} (-1)^p \binom{p}{q} \hat{f}_{n-p-1}(S_Q).$$

Now substitute $i = q$ and $k = n - p - 1$ in (5.6) and combine it with (5.9). \square

5.3. Ranks of $(\dot{E}_X)_{*,*}^1$. By construction of the modified spectral sequence, $(\dot{E}_X)_{p,q}^1 \cong (E_{\partial X})_{p,q}^2$ for $p \leq n - 1$. Let χ_q^2 be the Euler characteristic of q -th row of $(E_{\partial X})_{*,*}^2$:

$$(5.10) \quad \chi_q^2 = \sum_{p \leq n-1} (-1)^p \dim(E_{\partial X})_{p,q}^2.$$

Euler characteristics of the first and the second pages coincide: $\chi_q^2 = \chi_q^1$. By Theorem 1, for $q < p < n$ we have

$$\dim(\dot{E}_X)_{p,q}^1 = \binom{n}{q} \beta_p(S_Q).$$

Lemma 5.7 yields

$$(-1)^q \dim(\dot{E}_X)_{q,q}^1 + \sum_{p=q+1}^{n-1} (-1)^p \binom{n}{q} \beta_p(S_Q) = (\chi(S_Q) - 1) \binom{n}{q} + (-1)^q h_q(S_Q).$$

By taking into account the equality $\chi(S_Q) = \sum_{p=0}^{n-1} \beta_p(S_Q)$ and the obvious relation between reduced and non-reduced Betti numbers, this proves Theorem 2.

5.4. Manifold case. Now we prove Theorem 3. If Q is a Buchsbaum manifold with corners, then S_Q is a homology manifold. Then Poincare duality $\beta_i(S_Q) =$

$\beta_{n-1-i}(S_Q)$ and Dehn–Sommerville relations (5.5) imply

$$\begin{aligned}
 \dim(\dot{E}_X)_{q,q}^1 &= h_q + \binom{n}{q} \sum_{p=0}^q (-1)^{p+q} \tilde{\beta}_p = \\
 &= h_q - (-1)^q \binom{n}{q} + \binom{n}{q} \sum_{p=0}^q (-1)^{p+q} \beta_p = \\
 &= h_q - (-1)^q \binom{n}{q} + \binom{n}{q} \sum_{p=n-1-q}^{n-1} (-1)^{n-1-p+q} \beta_p = \\
 &= h_{n-q} + (-1)^q \binom{n}{q} \left[-(-1)^n + (-1)^n \chi + \sum_{p=n-1-q}^{n-1} (-1)^{n-1-p} \beta_p \right] = \\
 &= h_{n-q} + (-1)^q \binom{n}{q} \left[-(-1)^n + \sum_{p=0}^{n-q-2} (-1)^{p+n} \beta_p \right].
 \end{aligned}$$

The last expression in brackets coincides with $\sum_{p=-1}^{n-q-2} (-1)^{p+n} \tilde{\beta}_p$ whenever the summation is taken over nonempty set, that is for $q \leq n-2$. Thus $\dim(\dot{E}_X)_{q,q}^1 = h'_{n-q}$ for $q \leq n-2$. In the case $q = n-1$ we have $\dim(\dot{E}_X)_{n-1,n-1}^1 = h_1 + \binom{n}{n-1} = h'_1 + n$. This proves part (1) of Theorem 3.

Part (2) follows easily. Indeed, for $q = n$ we have

$$\dim(\dot{E}_X)_{n,n}^2 = \dim(\dot{E}_X)_{n,n}^1 = \binom{n}{n} \dim H_n(Q, \partial Q) = 1 = h'_0(S_Q)$$

For $q = n-1$:

$$\dim(\dot{E}_X)_{n-1,n-1}^2 = \dim(\dot{E}_X)_{n-1,n-1}^1 - \binom{n}{n-1} \dim \operatorname{Im} \delta_n = h'_1(S_Q),$$

since the map $\delta_n: H_n(Q, \partial Q) \rightarrow H_{n-1}(\partial Q)$ is injective and $\dim H_n(Q, \partial Q) = 1$.

If $q \leq n-2$, then $(\dot{E}_X)_{q,q}^2 = (\dot{E}_X)_{q,q}^1$, and the statement follows from part (1).

5.5. Cone case. If $P = P(S) \cong \operatorname{Cone} |S|$, then the map $\delta_i: H_i(P, \partial P) \rightarrow \tilde{H}_{i-1}(\partial P)$ is an isomorphism as follows from the long exact sequence of the pair $(P, \partial P)$. Thus for $q \leq n-1$, Theorem 1 implies

$$\dim(\dot{E}_X)_{q,q}^\infty = \dim(\dot{E}_X)_{q,q}^1 - \binom{n}{q} \dim H_{q+1}(P, \partial P) = \dim(\dot{E}_X)_{q,q}^1 - \binom{n}{q} \tilde{\beta}_q(S).$$

By Theorem 2 this expression is equal to

$$h_q + \binom{n}{q} \left[\sum_{p=0}^q (-1)^{p+q} \tilde{\beta}_p(S) \right] - \binom{n}{q} \tilde{\beta}_q(S) = h_q(S) + \binom{n}{q} \sum_{p=0}^{q-1} (-1)^{p+q} \tilde{\beta}_p(S) = h''_q.$$

The case $q = n$ follows from (5.3). Indeed, the term $(\dot{E}_X)_{n,n}^1$ survives in the spectral sequence, thus:

$$\dim(\dot{E}_X)_{n,n}^\infty = \binom{n}{n} \dim H_n(P, \partial P) = \beta_{n-1}(S) = h'_n(S) = h''_n(S).$$

This proves Theorem 4.

6. Homology of X .

In this section we suppose \mathbb{k} is a field. Theorem 1 gives an additional grading on $H_*(X)$, namely the one induced by degrees of exterior forms, as described below. In the following Q is an arbitrary Buchsbaum pseudo-cell complex of dimension n .

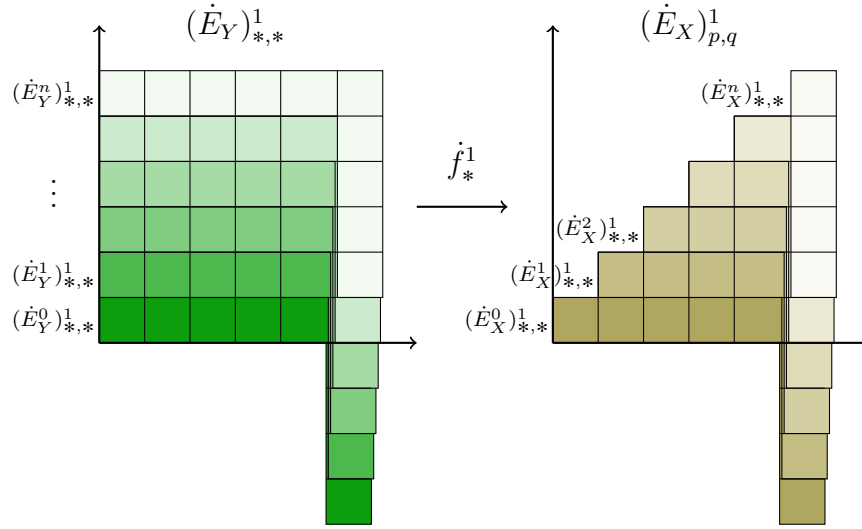


FIGURE 3. Decomposition of spectral sequences into graded components

CONSTRUCTION 6.1. The spectral sequence $(\dot{E}_Y)^*$ splits in the direct sum of spectral subsequences, indexed by degrees of exterior forms. For $0 \leq j \leq n$ consider the \neg -shaped spectral sequence

$$(\dot{E}_Y^j)_{p,q}^r = (\dot{E}_Q^j)_{p,q-j}^r \otimes \Lambda_j.$$

Clearly, $(\dot{E}_Y)_{*,*}^r = \bigoplus_{j=0}^n (\dot{E}_Y^j)_{*,*}^r$. This decomposition is sketched on Figure 3. Let $H_{i,j}(Y)$ denote the module $H_i(Q) \otimes \Lambda_j$. Then $(\dot{E}_Y^j)_{p,q}^r \Rightarrow H_{p+q-j,j}(Y)$.

Let us construct the corresponding \sqcap -shaped spectral subsequences in $(\dot{E}_X)_{*,*}^*$. Consider the bigraded vector subspaces $(\dot{E}_X^j)_{*,*}^1$:

$$(\dot{E}_X^j)_{p,q}^1 = \begin{cases} (\dot{E}_X)_{p,q}^1, & \text{if } q = j \text{ and } p < n; \\ 0, & \text{if } q \neq j \text{ and } p < n; \\ H_{q+n-j}(Q, \partial Q) \otimes \Lambda_j, & \text{if } p = n. \end{cases}$$

In the last case we used the isomorphism of Theorem 1. Theorem 1 implies that all differentials of $(\dot{E}_X)_{*,*}^*$ preserve the subspace $(\dot{E}_X^j)_{*,*}^*$, thus spectral subsequences $(\dot{E}_X^j)_{*,*}^r$ are well defined for $r \geq 2$, and $(\dot{E}_X)_{*,*}^r = \bigoplus_{j=0}^n (\dot{E}_X^j)_{*,*}^r$.

Over a field, $H_k(X)$ can be identified with the associated module $\bigoplus_{p+q=k} (E_X)_{p,q}^\infty$, and thus inherits a double grading:

$$H_k(X) \cong \bigoplus_{i+j=k} H_{i,j}(X),$$

where

$$H_{i,j}(X) \stackrel{\text{def}}{=} \bigoplus_{p+q=i+j} (\dot{E}_X^j)_{p,q}^\infty.$$

Thus we have $(\dot{E}_X^j)_{p,q}^r \Rightarrow H_{p+q-j,j}(X)$. The map $f_*^r: (\dot{E}_Y)^r \rightarrow (\dot{E}_X)^r$ sends $(\dot{E}_Y^j)^r$ to $(\dot{E}_X^j)^r$ for each $j \in \{0, \dots, n\}$. The map $f_*: H_*(Y) \rightarrow H_*(X)$ sends $H_{i,j}(Y)$ to $H_{i,j}(X)$.

THEOREM 5.

- (1) If $i > j$, then $f_*: H_{i,j}(Y) \rightarrow H_{i,j}(X)$ is an isomorphism. As a consequence, $H_{i,j}(X) \cong H_i(Q, \partial Q) \otimes \Lambda_j$.
- (2) If $i < j$, then there exists an isomorphism $H_{i,j}(X) \cong H_i(Q, \partial Q) \otimes \Lambda_j$.
- (3) In case $i = j < n$, the module $H_{i,i}(X)$ fits in the exact sequence

$$0 \rightarrow (\dot{E}_X)_{i,i}^\infty \rightarrow H_{i,i}(X) \rightarrow H_i(Q, \partial Q) \otimes \Lambda_i \rightarrow 0,$$

or, equivalently,

$$0 \rightarrow \text{Im } \delta_{i+1} \otimes \Lambda_i \rightarrow (\dot{E}_X)_{i,i}^1 \rightarrow H_{i,i}(X) \rightarrow H_i(Q, \partial Q) \otimes \Lambda_i \rightarrow 0$$

- (4) If $i = j = n$, then

$$H_{n,n}(X) = (\dot{E}_X)_{n,n}^\infty = (\dot{E}_X)_{n,n}^1 \cong H_n(Q, \partial Q).$$

PROOF. According to Theorem 1, the map $f_*^1: (\dot{E}_Y^j)_{i,q}^1 \rightarrow (\dot{E}_X^j)_{i,q}^1$ is an isomorphism if $i > j$ or $i = j = n$, and injective if $i = j$. For each j both spectral sequences

(\dot{E}_Y^j) and (\dot{E}_X^j) are \neg -shaped, thus fold in the long exact sequences:

$$(6.1) \quad \begin{array}{ccccccc} \dots & \longrightarrow & (\dot{E}_Y^j)_{i,j}^1 & \longrightarrow & H_{i,j}(Y) & \longrightarrow & (\dot{E}_Y^j)_{n,i-n+j}^1 \xrightarrow{(d_Y)^{n-i+1}} (\dot{E}_Y^j)_{i-1,j}^1 \longrightarrow \dots \\ & & \downarrow f_*^1 & & \downarrow f_* & & \cong \downarrow f_*^1 & & \downarrow f_* \\ \dots & \longrightarrow & (\dot{E}_X^j)_{i,j}^1 & \longrightarrow & H_{i,j}(X) & \longrightarrow & (\dot{E}_X^j)_{n,i-n+j}^1 \xrightarrow{(d_X)^{n-i+1}} (\dot{E}_X^j)_{i-1,j}^1 \longrightarrow \dots \end{array}$$

Application of five lemma in the case $i > j$ proves (1). For $i < j$, the groups $(\dot{E}_X^j)_{i,j}^1, (\dot{E}_X^j)_{i-1,j}^1$ vanish by dimensional reasons, thus $H_{i,j}(X) \cong (\dot{E}_X^j)_{n,i-n+j}^1 \cong (\dot{E}_Y^j)_{n,i-n+j}^1 \cong H_i(Q, \partial Q) \otimes \Lambda_j$. Case $i = j$ also follows from (6.1) by a simple diagram chase. \square

In case of manifolds Theorem 5 reveals a bigraded duality. If Q is a nice manifold with corners, $Y = Q \times T^n$, and λ is a characteristic function over \mathbb{Z} , then $X = Y/\sim$ is a compact orientable topological manifold with the locally standard torus action. In this case Poincare duality respects the double grading.

PROPOSITION 6.2. *Let Q be a Buchsbaum manifold with corners and X is a quotient construction over Q determined by a \mathbb{Z} -characteristic function. Then $H_{i,j}(X; \mathbb{k}) \cong H_{n-i,n-j}(X; \mathbb{k})$ for any field \mathbb{k} .*

PROOF. When $i < j$, we have

$$H_{i,j}(X) \cong H_i(Q, \partial Q) \otimes \Lambda_j \cong H_{n-i}(Q) \otimes \Lambda_{n-j} \cong H_{n-i,n-j}(X),$$

by the Poincare–Lefschetz duality applied to Q and Poincare duality applied to torus. The remaining isomorphism $H_{i,i}(X) \cong H_{n-i,n-i}(X)$ now follows from the ordinary Poincare duality in X . \square

REMARK 6.3. If X is determined by \mathbb{Q} -characteristic function, then it is a homology \mathbb{Q} -manifold. In this case Proposition 6.2 holds over \mathbb{Q} .

7. One example with non-acyclic proper faces

Let Q be the product of S^1 with the closed interval $\mathbb{I} = [-1, 1] \subset \mathbb{R}^1$. Then Q is a nice manifold with corners having two proper faces: $F_1 = S^1 \times \{-1\}$ and $F_2 = S^1 \times \{1\}$. The faces are not acyclic, so the arguments of the paper cannot be applied. Consider the 2-torus T^2 with a given coordinate splitting $T^2 = T^{\{\{1\}\}} \times T^{\{\{2\}\}}$.

First, define the characteristic function λ on Q by

$$\lambda(F_1) = T^{\{\{1\}\}}, \quad \lambda(F_2) = T^{\{\{2\}\}}.$$

The corresponding manifold with locally standard action is

$$X = (S^1 \times \mathbb{I} \times T^2)/\sim = S^1 \times (\mathbb{I} \times T^2/\sim) = S^1 \times \mathcal{Z}_{\mathbb{I}} \cong S^1 \times S^3.$$

Here $\mathcal{Z}_{\mathbb{I}}$ is the moment-angle manifold of the interval \mathbb{I} , see [5] or [7].

Next, consider the characteristic function λ' on Q determined by

$$\lambda'(F_1) = \lambda'(F_2) = T^{\{\{1\}\}}.$$

The corresponding manifold is

$$X' = (S^1 \times \mathbb{I} \times T^2)/\sim = S^1 \times T^{\{\{2\}\}} \times (\mathbb{I} \times T^{\{\{1\}\}}/\sim) \cong S^1 \times S^1 \times S^2.$$

This example shows that in general Betti numbers of $(Q \times T^2)/\sim$ depend not only on Q , but may also depend on the characteristic function. This is opposite to the situation when proper faces are acyclic, as was shown in the paper.

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